

# A DUAL-PRIMAL COUPLING TECHNIQUE WITH LOCAL TIME STEP FOR WAVE PROPAGATION PROBLEMS

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**Abstract.** *We are interested in space-time refinement methods for linear wave propagation. In<sup>1,2</sup>, some stable numerical schemes using non-conforming grids in space and time have been proposed. These methods use a Lagrange multiplier to cope with the interface conditions. The choice of the discretization space of this additional unknown can be in some cases not trivial.*

*In the present paper we propose an alternative method. The main new idea is to use different variational formulations in the fine and in the coarse grids. We present a time discretization that leads to the conservation of a discrete energy and provide a complete stability and error analysis in the case where the time step is twice smaller in one domain than in the other one.*

## 1 INTRODUCTION

When coupling two wave propagation problems in two different regions, it can be interesting, in order to keep the same accuracy in the whole domain, to introduce non matching grids in space. Concerning the time discretization, with explicit schemes, when a uniform time step is used, it must be chosen in such a way that both CFL stability conditions for each region are satisfied, which can be very restrictive in one of the domains. Furthermore, it can give rise to large dispersion errors in the region where the time step is far from the largest value that ensures the stability. A natural idea is then to introduce a local time step in order to reduce the computational cost and to improve the accuracy.

In<sup>1,2</sup>, the authors have proposed methods allowing to use non conforming meshes in space and in time. The space discretization is performed using a Lagrange multiplier in order to ensure the transmission conditions. The time stepping is based on finite differences and on a discrete energy conservation which gives a robust numerical method. Following the same ideas, we propose an alternative method that avoids the use of a Lagrange multiplier by means of using a so-called “primal-dual formulation”. The time discretization is based as the previous ones, on a discrete energy conservation and therefore is stable by construction. This method has been developed in an abstract framework and can be applied to several situations as space-time mesh refinements in acoustics, elastodynamics and electromagnetics, and also to some

situations where the equations are not the same in different regions of the domain, as for instance fluid-structure interaction problems.

In this paper, we describe the method in a general setting. We give a stability result and present the error estimates for the 2D and 3D problems in the case where the time step in one region is twice smaller than the one in the other region. The main result concerns an  $L^2$  error estimate in  $O(h^{3/2})$  where  $h$  denotes the space discretization step. The proof is based on energy techniques and boot-strap arguments, as the one given in<sup>3</sup> for the method with Lagrange multiplier in 1D. We would like to note that, with the new method, the absence of Lagrange multiplier simplifies the proof and allows us to obtain the convergence for any dimension.

## 2 An abstract variational formulation

This paper deals with the simulation of coupled linear wave propagation problems. Let us assume that the computational domain is composed by two sub-domains  $\Omega_l$ ,  $l \in \{c, f\}$ . We denote by  $(u_l, p_l) \in \mathbb{R}^{m_l} \times \mathbb{R}^{n_l}$ ,  $l \in \{c, f\}$  the unknowns of the problem that will satisfy the following equations

$$\begin{cases} \mathcal{A}_l \frac{\partial u_l}{\partial t} + \mathcal{D}_l^* p_l = g_l, \\ \mathcal{B}_l \frac{\partial p_l}{\partial t} - \mathcal{D}_l u_l = f_l. \end{cases} \quad (1)$$

We assume that  $\mathcal{A}_l$  and  $\mathcal{B}_l$  are symmetric matrices depending on the space variable and that  $\mathcal{D}_l$  and  $\mathcal{D}_l^*$  are first order differential operators in space that are adjoint in the distributional sense, that is, for all functions  $\varphi_l$  and  $\psi_l$  that are regular enough and with compact support in  $\Omega_l$  we have

$$\int_{\Omega_l} \mathcal{D}_l \varphi \psi \, dx = \int_{\Omega_l} \mathcal{D}_l^* \psi \varphi \, dx.$$

Both systems are coupled by the continuity of the generalized traces of the solution through the interface  $\Gamma = \overline{\Omega_c} \cap \overline{\Omega_f}$  that will depend on the nature of the problem. In section 3 some examples will be given.

We introduce the Hilbert spaces

$$H_l := [L^2(\Omega_l)]^{m_l}, \quad V_l := [L^2(\Omega_l)]^{n_l}, \quad l \in \{c, f\},$$

equipped with the usual norms and scalar products. We also define the Hilbert spaces

$$X_l := \{ \tilde{u}_l \in H_l / \mathcal{D}_l u_l \in V_l \}, \quad , l \in \{c, f\},$$

and their natural norms and scalar products

$$\begin{cases} (u_l, \tilde{u}_l)_{X_l} := (u_l, \tilde{u}_l)_{H_l} + (\mathcal{D}_l u_l, \mathcal{D}_l \tilde{u}_l)_{V_l}, & \forall (u_l, \tilde{u}_l) \in X_l \times X_l, \\ \|\tilde{u}_l\|_{X_l}^2 := \|\tilde{u}_l\|_{H_l}^2 + \|\mathcal{D}_l \tilde{u}_l\|_{V_l}^2, & \forall \tilde{u}_l \in X_l. \end{cases} \quad (2)$$

We introduce the following product spaces

$$H := H_c \times H_f, \quad V := V_c \times V_f, \quad X := X_c \times X_f,$$

equipped with the norms

$$\begin{cases} \|(u_c, u_f)\|_H^2 &:= \|u_c\|_{H_c}^2 + \|u_f\|_{H_f}^2, \quad \forall (u_c, u_f) \in H_c \times H_f, \\ \|(p_c, p_f)\|_V^2 &:= \|p_c\|_{V_c}^2 + \|p_f\|_{V_f}^2, \quad \forall (p_c, p_f) \in V_c \times V_f, \\ \|(u_c, u_f)\|_X^2 &:= \|u_c\|_{X_c}^2 + \|u_f\|_{X_f}^2, \quad \forall (u_c, u_f) \in X_c \times X_f. \end{cases}$$

In such a way, the abstract variational formulation that we deal with is

$$\begin{cases} \text{Find } (u_c, u_f) \in C^1([0, T]; H) \cap C^0([0, T]; X) \text{ and } (p_c, p_f) \in C^1([0, T]; V) \text{ such that} \\ \frac{d}{dt}(\mathcal{A}_c u_c, \tilde{u}_c)_{H_c} + (\mathcal{D}_c \tilde{u}_c, p_c)_{V_c} - c(\tilde{u}_c, u_f) = (g_c, \tilde{u}_c)_{X_c' \times X_c}, \quad \forall \tilde{u}_c \in X_c, \\ \frac{d}{dt}(\mathcal{B}_c p_c, \tilde{p}_c)_{V_c} - (\mathcal{D}_c u_c, \tilde{p}_c)_{V_c} = (f_c, \tilde{p}_c)_{V_c' \times V_c}, \quad \forall \tilde{p}_c \in V_c, \\ \frac{d}{dt}(\mathcal{A}_f u_f, \tilde{u}_f)_{H_f} + (\mathcal{D}_f \tilde{u}_f, p_f)_{V_f} + c(u_c, \tilde{u}_f) = (g_f, \tilde{u}_f)_{X_f' \times X_f}, \quad \forall \tilde{u}_f \in X_f, \\ \frac{d}{dt}(\mathcal{B}_f p_f, \tilde{p}_f)_{V_f} - (\mathcal{D}_f u_f, \tilde{p}_f)_{V_f} = (f_f, \tilde{p}_f)_{V_f' \times V_f}, \quad \forall \tilde{p}_f \in V_f, \end{cases} \quad (3)$$

system that must be completed with the initial and boundary conditions that we will omit for the sake of simplicity. The bilinear operator

$$\begin{cases} c : X_c \times X_f \longrightarrow \mathbb{R} \\ (\tilde{u}_c, \tilde{u}_f) \longmapsto c(\tilde{u}_c, \tilde{u}_f), \end{cases}$$

couples the formulations on each domain and is supposed to be continuous, that is,

$$c(\tilde{u}_c, \tilde{u}_f) \leq \|c\| \|\tilde{u}_c\|_{X_c} \|\tilde{u}_f\|_{X_f}, \quad \forall (u_c, u_f) \in X_c \times X_f.$$

We will assume that the operators defined by the symmetric matrices  $\mathcal{A}_l$  and  $\mathcal{B}_l$  belong to  $\mathcal{L}(H_l, H_l)$  and  $\mathcal{L}(V_l, V_l)$  respectively and that there exists a constant  $\mathcal{C}$  such that

$$\begin{cases} \forall \tilde{u}_l \in H_l, \quad (\mathcal{A}_l \tilde{u}_l, \tilde{u}_l)_{H_l} \geq \mathcal{C} \|\tilde{u}_l\|_{H_l}^2, \\ \forall \tilde{p}_l \in V_l, \quad (\mathcal{B}_l \tilde{p}_l, \tilde{p}_l)_{V_l} \geq \mathcal{C} \|\tilde{p}_l\|_{V_l}^2. \end{cases}$$

**Remark 2.1** If we take  $(\tilde{u}_c, \tilde{u}_f) = (u_c, u_f)$  and  $(\tilde{p}_c, \tilde{p}_f) = (p_c, p_f)$  in the equations (3) we obtain

$$\frac{d\mathbf{E}}{dt} = \sum_{l \in \{c, f\}} \{ (g_l, u_l)_{X_c' \times X_c} + (f_l, p_l)_{V_c' \times V_c} \}$$

with

$$\begin{cases} \mathbf{E}(t) &= \mathbf{E}_c(t) + \mathbf{E}_f(t), \\ \mathbf{E}_l(t) &= \frac{1}{2} ((\mathcal{A}_l u_l, u_l)_{H_l} + (\mathcal{B}_l p_l, p_l)_{V_l}), \quad l \in \{c, f\}. \end{cases}$$

In particular, in absence of external forces, the energy that we have defined above is conserved.

**Remark 2.2** The functions in the space  $X_l$ ,  $l \in \{c, f\}$  have more regularity in space than the functions in the space  $V_l$ ,  $l \in \{c, f\}$ . We point out that the coupling of both system, that is done with the operator  $c(\cdot, \cdot)$ , makes appear only the unknowns  $u_l$ ,  $l \in \{c, f\}$  that we will call the regular variables.

### 3 Some examples of application

#### 1. Mesh refinement for acoustics in dimension $d$ :

$$\begin{cases} u_c &\equiv \underline{v}_c, & (\text{the velocity field on } \Omega_c, m_c = d), \\ p_c &\equiv p_c, & (\text{the pressure on } \Omega_c, n_c = 1), \\ \mathcal{A}_c &\equiv \underline{\underline{A}}_c, & (\text{the anisotropy tensor on } \Omega_c), \\ \mathcal{B}_c &\equiv \rho_c, & (\text{the density of the fluid on } \Omega_c), \\ \mathcal{D}_c &\equiv \text{div}(\cdot), \quad \mathcal{D}_c^* &\equiv -\underline{\nabla}(\cdot), \end{cases} \quad (4)$$

$$\begin{cases} u_f &\equiv p_f, & (\text{the pressure on } \Omega_f, m_f = 1), \\ p_f &\equiv \underline{v}_f, & (\text{the velocity field on } \Omega_f, n_f = d), \\ \mathcal{A}_f &\equiv \rho_f, & (\text{the density of the fluid on } \Omega_f), \\ \mathcal{B}_f &\equiv \underline{\underline{A}}_f, & (\text{the anisotropy tensor on } \Omega_f), \\ \mathcal{D}_f &\equiv \underline{\nabla}(\cdot), \quad \mathcal{D}_f^* &\equiv -\text{div}(\cdot), \end{cases} \quad (5)$$

$$c(u_c, u_f) \equiv \langle \underline{v}_c \cdot \underline{n}, p_f \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)}. \quad (6)$$

#### 2. Mesh refinement for the 3-D Maxwell's equations:

$$\begin{cases} u_c &\equiv \underline{E}_c, & (\text{the electric field on } \Omega_c, m_c = 3), \\ p_c &\equiv \underline{H}_c, & (\text{the magnetic field on } \Omega_c, n_c = 3), \\ \mathcal{A}_c &\equiv \varepsilon_c, & (\text{the dielectric permittivity on } \Omega_c), \\ \mathcal{B}_c &\equiv \mu_c, & (\text{the magnetic permeability on } \Omega_c), \\ \mathcal{D}_c &\equiv -\underline{\text{rot}}(\cdot), \quad \mathcal{D}_c^* &\equiv -\underline{\text{rot}}(\cdot), \end{cases} \quad (7)$$

$$\left\{ \begin{array}{ll} u_f & \equiv \underline{H}_f, & (\text{the magnetic field on } \Omega_f, m_f = 3), \\ p_f & \equiv \underline{E}_f, & (\text{the electric field on } \Omega_f, n_f = 3), \\ \mathcal{A}_f & \equiv \mu_f, & (\text{the magnetic permeability on } \Omega_f), \\ \mathcal{B}_f & \equiv \varepsilon_f, & (\text{the electric permittivity on } \Omega_f), \\ \mathcal{D}_f & \equiv \underline{\text{rot}}(\cdot), & \mathcal{D}_f^* \equiv \underline{\text{rot}}^*(\cdot), \end{array} \right. \quad (8)$$

$$c(u_c, u_f) \equiv \langle \underline{H}_f \wedge \underline{n}, \underline{n} \wedge (\underline{E}_c \wedge \underline{n}) \rangle_{\underline{H}_{\parallel}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma), \underline{H}_{\perp}^{-\frac{1}{2}}(\text{rot}_{\Gamma}, \Gamma)}, \quad (9)$$

where the space  $\underline{H}_{\parallel}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma)$  and its dual  $\underline{H}_{\perp}^{-\frac{1}{2}}(\text{rot}_{\Gamma}, \Gamma)$  are defined in<sup>4</sup>. We point out that this application is specially interesting because the primal and the dual formulations are almost the same.

3. *Mesh refinement for elastodynamics in dimension  $d$ .* In this case we must change the spaces  $H_c$  and  $V_f$  in order to impose the symmetry of the stress tensor:

$$\left\{ \begin{array}{l} H_c := \left\{ \underline{\underline{\sigma}}_c \in [L^2(\Omega_c)]^{d \times d} / \underline{\underline{\sigma}}_c \text{ is symmetric} \right\}, \\ V_f := \left\{ \underline{\underline{\sigma}}_f \in [L^2(\Omega_f)]^{d \times d} / \underline{\underline{\sigma}}_f \text{ is symmetric} \right\}. \end{array} \right. \quad (10)$$

In this way we have

$$\left\{ \begin{array}{ll} u_c & \equiv \underline{\underline{\sigma}}_c, & (\text{the stress tensor on } \Omega_c, m_c = d^2 (= d \times d)), \\ p_c & \equiv \underline{v}_c, & (\text{the velocity field on } \Omega_c, n_c = d), \\ \mathcal{A}_c & \equiv \mathbf{A}_c, & (\text{the inverse of the elasticity tensor on } \Omega_c), \\ \mathcal{B}_c & \equiv \rho_c, & (\text{the density of the material on } \Omega_c), \\ \mathcal{D}_c & \equiv \underline{\underline{\text{div}}}(\cdot), & \mathcal{D}_c^* \equiv -\underline{\underline{\varepsilon}}(\cdot), \end{array} \right. \quad (11)$$

$$\left\{ \begin{array}{ll} u_f & \equiv \underline{v}_f, & (\text{the velocity field on } \Omega_f, m_f = d), \\ p_f & \equiv \underline{\underline{\sigma}}_f, & (\text{the stress tensor on } \Omega_f, n_f = d^2 (= d \times d)), \\ \mathcal{A}_f & \equiv \rho_f, & (\text{the density of the material on } \Omega_f), \\ \mathcal{B}_f & \equiv \mathbf{A}_f, & (\text{the inverse of the elasticity tensor on } \Omega_f), \\ \mathcal{D}_f & \equiv \underline{\underline{\varepsilon}}(\cdot), & \mathcal{D}_f^* \equiv -\underline{\underline{\text{div}}}(\cdot), \end{array} \right. \quad (12)$$

$$c(u_c, u_f) \equiv \langle \underline{\underline{\sigma}}_c \underline{n}, \underline{v}_f \rangle_{\underline{H}^{-\frac{1}{2}}(\Gamma), \underline{H}^{\frac{1}{2}}(\Gamma)}. \quad (13)$$

**Remark 3.1** *The symmetry in the spaces  $H_c$  and  $V_f$  must be imposed in order to ensure that the  $\underline{\underline{\varepsilon}}(\cdot)$  is the adjoint  $\underline{\underline{\text{div}}}(\cdot)$  and the invertibility of the elasticity tensor.*

4. *Fluid-solid interaction in dimension  $d$ .* In this case we must also impose the symmetry in the space  $V_f$  given in (10). We have then

$$\left\{ \begin{array}{ll} u_c \equiv p_c, & \text{(the pressure of the fluid, } m_c = 1), \\ p_c \equiv \underline{v}_c, & \text{(the velocity field of the fluid, } n_c = d), \\ \mathcal{A}_c \equiv \rho_c, & \text{(the density of the fluid),} \\ \mathcal{B}_c \equiv \underline{\underline{A}}_c, & \text{(the anisotropy tensor),} \\ \mathcal{D}_c \equiv \underline{\nabla}(\cdot), & \mathcal{D}_c^* \equiv -\text{div}(\cdot), \end{array} \right. \quad (14)$$

$$\left\{ \begin{array}{ll} u_f \equiv \underline{v}_f, & \text{(the velocity field on the solid, } m_f = d), \\ p_f \equiv \underline{\underline{\sigma}}_f, & \text{(the stress tensor on the solid, } n_f = d^2 (= d \times d)), \\ \mathcal{A}_f \equiv \rho_f, & \text{(the density of the material),} \\ \mathcal{B}_f \equiv \mathbf{A}_f, & \text{(the inverse of the elasticity tensor),} \\ \mathcal{D}_f \equiv \underline{\underline{\varepsilon}}(\cdot), & \mathcal{D}_f^* \equiv -\underline{\underline{\text{div}}}(\cdot), \end{array} \right. \quad (15)$$

$$c(u_c, u_f) \equiv \int_{\Gamma} \underline{v}_f \cdot \underline{n} p_c \, d\gamma. \quad (16)$$

For numerical results obtained with this method for fluid-structure interaction, the reader can refer to<sup>5</sup>.

In all examples, the continuity of the operator  $c(\cdot, \cdot)$  can be shown using traces theorems adapted for each situation.

#### 4 Space discretization

For the space discretization of the problem (3) we follow a Galerkin approach. In this way we construct finite dimensional spaces

$$\left| \begin{array}{ll} X_{l,h} \subset X_l, & V_{l,h} \subset V_l, \\ X_h := X_{c,h} \times X_{f,h}, & V_h := V_{c,h} \times V_{f,h}, \end{array} \right. \quad l \in \{c, f\},$$

( $h$  is a discretization parameter) satisfying the usual approximation properties

$$\left| \begin{array}{l} \lim_{h \rightarrow 0} \inf_{\tilde{u}_l^h \in X_{l,h}} \|u_l - \tilde{u}_l^h\|_{X_l} = 0, \quad \forall u_l \in X_l, \\ \lim_{h \rightarrow 0} \inf_{\tilde{p}_l^h \in V_{l,h}} \|p_l - \tilde{p}_l^h\|_{V_l} = 0, \quad \forall p_l \in V_l, \end{array} \right|, \quad l \in \{c, f\}.$$

The semi-discrete problem is then

$$\left| \begin{array}{l} \text{Find } (u_c^h, u_f^h) \in C^1([0, T]; X_h) \text{ et } (p_c^h, p_f^h) \in C^1([0, T]; V_h) \text{ such that} \\ \frac{d}{dt}(\mathcal{A}_c u_c^h, \tilde{u}_c^h)_{H_c} + (\mathcal{D}_c \tilde{u}_c^h, p_c^h)_{V_c} - c(\tilde{u}_c^h, u_f^h) = (g_c, \tilde{u}_c^h)_{X'_c \times X_c}, \\ \frac{d}{dt}(\mathcal{B}_c p_c^h, \tilde{p}_c^h)_{V_c} - (\mathcal{D}_c u_c^h, \tilde{p}_c^h)_{V_c} = (f_c, \tilde{p}_c^h)_{V'_c \times V_c}, \\ \frac{d}{dt}(\mathcal{A}_f u_f^h, \tilde{u}_f^h)_{H_f} + (\mathcal{D}_f \tilde{u}_f^h, p_f^h)_{V_f} + c(u_c^h, \tilde{u}_f^h) = (g_f, \tilde{u}_f^h)_{X'_f \times X_f}, \\ \frac{d}{dt}(\mathcal{B}_f p_f^h, \tilde{p}_f^h)_{V_f} - (\mathcal{D}_f u_f^h, \tilde{p}_f^h)_{V_f} = (f_f, \tilde{p}_f^h)_{V'_f \times V_f}, \\ \forall ((\tilde{u}_c^h, \tilde{u}_f^h), (\tilde{p}_c^h, \tilde{p}_f^h)) \in X_h \times V_h. \end{array} \right. \quad (17)$$

**Remark 4.1** *It is clear that in absence of external forces, the semi-discrete energy*

$$\left| \begin{array}{l} \mathbf{E}_h(t) = \mathbf{E}_{c,h}(t) + \mathbf{E}_{f,h}(t), \\ \mathbf{E}_{l,h}(t) = \frac{1}{2} ((\mathcal{A}_l u_l^h, u_l^h)_{H_l} + (\mathcal{B}_l p_l^h, p_l^h)_{V_l}), \quad l \in \{c, f\}, \end{array} \right. \quad (18)$$

*is conserved.*

## 5 Time stepping

**The interior scheme.** A second order finite difference scheme that computes  $u_l^h$  and  $v_l^h$  in different time steps is used for the time discretization. As it has been explained before, we will use a time step  $2\Delta t$  on the domain  $\Omega_c$  that is twice larger than the one on  $\Omega_f$ . In this way, we have

$$\left| \begin{array}{l} \text{Find } (u_c^{2n}, p_c^{2n+1}, u_f^n, p_f^{n+\frac{1}{2}}) \in X_{c,h} \times V_{c,h} \times X_{f,h} \times V_{f,h} \text{ such that} \\ (\mathcal{A}_c \frac{u_c^{2n+2} - u_c^{2n}}{2\Delta t}, \tilde{u}_c^h)_{H_c} + (\mathcal{D}_c \tilde{u}_c^h, p_c^{2n+1})_{V_c} - c(\tilde{u}_c^h, \llbracket u_f \rrbracket^{2n+1}) = (g_c^{2n+1}, \tilde{u}_c^h)_{X'_c \times X_c}, \\ (\mathcal{B}_c \frac{p_c^{2n+1} - p_c^{2n-1}}{2\Delta t}, \tilde{p}_c^h)_{V_c} - (\mathcal{D}_c u_c^{2n}, \tilde{p}_c^h)_{V_c} = (f_c^{2n}, \tilde{p}_c^h)_{V'_c \times V_c}, \\ (\mathcal{A}_f \frac{u_f^{n+1} - u_f^n}{\Delta t}, \tilde{u}_f^h)_{H_f} + (\mathcal{D}_f \tilde{u}_f^h, p_f^{n+\frac{1}{2}})_{V_f} + c(\llbracket u_c \rrbracket^{n+\frac{1}{2}}, \tilde{u}_f^h) = (g_f^{n+\frac{1}{2}}, \tilde{u}_f^h)_{X'_f \times X_f}, \\ (\mathcal{B}_f \frac{p_f^{n+\frac{1}{2}} - p_f^{n-\frac{1}{2}}}{\Delta t}, \tilde{p}_f^h)_{V_f} - (\mathcal{D}_f u_f^n, \tilde{p}_f^h)_{V_f} = (f_f^n, \tilde{p}_f^h)_{V'_f \times V_f}, \\ \forall ((\tilde{u}_c^h, \tilde{u}_f^h), (\tilde{p}_c^h, \tilde{p}_f^h)) \in X_h \times V_h. \end{array} \right. \quad (19)$$

where  $\llbracket u_f \rrbracket^{2n+1}$  and  $\llbracket u_c \rrbracket^{n+\frac{1}{2}}$  are approximations of  $u_f^h(t^{2n+1})$  and  $u_c^h(n + \frac{1}{2})$  respectively that must be determined. We point out that for each time interval  $[t^{2n}, t^{2n+2}]$  there are three of these quantities. We must then write three linear independent equations that will allow us to obtain them and to couple both systems.

**The coupling equations: Energy conservation** The additional equations that we will add in order to couple the two systems in (19) will be chosen in such a way that the stability of the scheme will be ensured a priori. A simple way to do that is to impose a discrete version of the energy conservation property explained on the remarks 2.1 and 4.1. We introduce the discrete energy at the even time steps by

$$\mathbf{E}^{2n} := \mathbf{E}_c^{2n} + \mathbf{E}_f^{2n}, \quad (20)$$

where

$$\begin{aligned} \mathbf{E}_c^{2n} &:= \frac{1}{2}((\mathcal{A}_c u_c^{2n}, u_c^{2n})_{H_c} + (\mathcal{B}_c p_c^{2n+1}, p_c^{2n-1})_{V_c}), \\ \mathbf{E}_f^{2n} &:= \frac{1}{2}((\mathcal{A}_f u_f^n, u_f^n)_{H_f} + (\mathcal{B}_f p_f^{n+\frac{1}{2}}, p_f^{n-\frac{1}{2}})_{V_f}), \end{aligned}$$

and so we can establish the

**Theorem 5.1 (Conservative scheme)** *In order to complete the interior scheme (19) and obtain a numerical scheme that conserves the energy (20) in absence of external forces, the additional equations must be compatible with the following equality*

$$c(\llbracket u_c \rrbracket^{2n+\frac{1}{2}}, \frac{u_f^{2n} + u_f^{2n+1}}{4}) + c(\llbracket u_c \rrbracket^{2n+\frac{3}{2}}, \frac{u_f^{2n+1} + u_f^{2n+2}}{4}) = c(\frac{u_c^{2n+2} + u_c^{2n}}{2}, \llbracket u_f \rrbracket^{2n+1}). \quad (21)$$

**Proof:** We introduce in the first equation of (19) the test function  $\tilde{u}^h = (u_c^{2n+2} + u_c^{2n})/2$  to obtain

$$\left| \begin{aligned} &\frac{1}{4\Delta t} ((\mathcal{A}_c u_c^{2n+2}, u_c^{2n+2})_{H_c} - (\mathcal{A}_c u_c^{2n}, u_c^{2n})_{H_c}) = \\ &\quad - (\mathcal{D}_c \frac{u_c^{2n+2} + u_c^{2n}}{2}, p_c^{2n+1})_{V_c} + c(\frac{u_c^{2n+2} + u_c^{2n}}{2}, \llbracket u_f \rrbracket^{2n+1}). \end{aligned} \right. \quad (22)$$

Using the second equation on (19) for two successive time steps we have

$$(\mathcal{B}_c \frac{p_c^{2n+3} - p_c^{2n-1}}{4\Delta t}, \tilde{p}_c^h)_{V_c} = (\mathcal{D}_c \frac{u_c^{2n+2} + u_c^{2n}}{2}, \tilde{p}_c^h)_{V_c},$$

and then, taking  $\tilde{p}_c^h = p_c^{2n+1}$ ,

$$\frac{1}{4\Delta t} ((\mathcal{B}_c p_c^{2n+3}, p_c^{2n+1})_{V_c} - (\mathcal{B}_c p_c^{2n+1}, p_c^{2n-1})_{V_c}) = (\mathcal{D}_c \frac{u_c^{2n+2} + u_c^{2n}}{2}, p_c^{2n-1})_{V_c}. \quad (23)$$



We add the equations (22) and (23) to conclude that

$$\frac{1}{2\Delta t} (\mathbf{E}_c^{2n+2} - \mathbf{E}_c^{2n}) = c \left( \frac{u_c^{2n+2} + u_c^{2n}}{2}, \llbracket u_f \rrbracket^{2n+1} \right). \quad (24)$$

Using similar techniques it is easy to show that

$$\frac{1}{\Delta t} (\mathbf{E}_f^{n+1} - \mathbf{E}_f^n) = c \left( \llbracket u_c \rrbracket^{n+\frac{1}{2}}, \frac{u_f^n + u_f^{n+1}}{2} \right). \quad (25)$$

We conclude the proof combining (24) and (25) and the definition of the total discrete energy (20).  $\square$

We propose then the following equations in order to complete (19) and conserve the energy

$$\left\{ \begin{array}{l} \llbracket u_f \rrbracket^{2n+1} := \frac{u_f^{2n} + 2u_f^{2n+1} + u_f^{2n+2}}{4} \\ \llbracket u_c \rrbracket^{2n+\frac{1}{2}} := \frac{u_c^{2n+2} + u_c^{2n}}{2}, \quad \llbracket u_c \rrbracket^{2n+\frac{3}{2}} := \frac{u_c^{2n+2} + u_c^{2n}}{2}. \end{array} \right. \quad (26)$$

**Corollary 5.1** *The solution of the scheme (19)-(26), in absence of external forces, satisfies the conservation of the discrete energy at even time steps:*

$$\mathbf{E}^{2n} = \mathbf{E}^0, \quad \forall n \geq 0 \quad (27)$$

## 6 Stability of the scheme

We define for  $l \in \{c, f\}$

$$\|D_{l,h}\| = \sup_{u_l^h, p_l^h} \frac{(\mathcal{D}_l u_l^h, p_l^h)_{V_l}}{(\mathcal{A}_l u_l^h, u_l^h)_{H_l}^{1/2} (\mathcal{B}_l p_l^h, p_l^h)_{V_l}^{1/2}} \quad (28)$$

**Proposition 6.1 (Stability)** *Assume that:*

- the external forces are zero,
- $\mathbf{E}^0$  is bounded by a constant independent of the discretization steps,
- there exists a constant  $\Upsilon > 1$  independent of the discretization steps, such that

$$\|D_{c,h}\| \Delta t \leq \sqrt{1 - \frac{1}{\Upsilon^2}}, \quad \frac{\|D_{f,h}\| \Delta t}{2} \leq \sqrt{1 - \frac{1}{\Upsilon^2}} \quad (29)$$

Then, there exists a constant  $\mathcal{C}$  independent of the discretization steps such that  $(u_c^{2n}, p_c^{2n+1}, u_f^n, p_f^{n+\frac{1}{2}})$  solution of (19)–(26) satisfies

$$\left\{ \begin{array}{l} \|u_c^{2n}\|_{H_c} + \|p_c^{2n+1}\|_{V_c} \leq \mathcal{C} \Upsilon \sqrt{\mathbf{E}^0}, \\ \|u_f^n\|_{H_f} + \|p_f^{n+\frac{1}{2}}\|_{V_f} \leq \mathcal{C} \Upsilon \sqrt{\mathbf{E}^0}. \end{array} \right. \quad (30)$$

This implies that the numerical scheme is stable.

**Proof:** We recall that the numerical scheme has been constructed in such a way that, in absence of external forces, the discrete energy is conserved, (27). We will show that this quantity is equivalent to the  $L^2$  norm of the solution to complete the proof. Using the symmetry of the matrix  $\mathcal{B}_c$  we obtain that

$$\begin{aligned} (\mathcal{B}_c p_c^{2n+1}, p_c^{2n-1})_{V_c} &= (\mathcal{B}_c \frac{p_c^{2n+1} + p_c^{2n-1}}{2}, \frac{p_c^{2n+1} + p_c^{2n-1}}{2})_{V_c} - \\ &\quad (\mathcal{B}_c \frac{p_c^{2n+1} - p_c^{2n-1}}{2}, \frac{p_c^{2n+1} - p_c^{2n-1}}{2})_{V_c}. \end{aligned} \quad (31)$$

Now, with the second equation of (19) and the first inequality in (29) we have

$$\begin{aligned} (\mathcal{B}_c \frac{p_c^{2n+1} - p_c^{2n-1}}{2}, \frac{p_c^{2n+1} - p_c^{2n-1}}{2})_{V_c} &= \Delta t (\mathcal{D}_c u_c^{2n}, \frac{p_c^{2n+1} - p_c^{2n-1}}{2})_{V_c} \\ &\leq \left(1 - \frac{1}{\Upsilon^2}\right)^{\frac{1}{2}} \left(\mathcal{B}_c \frac{p_c^{2n+1} - p_c^{2n-1}}{2}, \frac{p_c^{2n+1} - p_c^{2n-1}}{2}\right)_{V_c}^{\frac{1}{2}} (\mathcal{A}_c u_c^{2n}, u_c^{2n})_{H_c}^{\frac{1}{2}}, \end{aligned}$$

that implies

$$(\mathcal{B}_c \frac{p_c^{2n+1} - p_c^{2n-1}}{2}, \frac{p_c^{2n+1} - p_c^{2n-1}}{2})_{V_c} \leq \left(1 - \frac{1}{\Upsilon^2}\right) (\mathcal{A}_c u_c^{2n}, u_c^{2n})_{H_c}.$$

Using this last inequality on (31) we easily obtain that

$$2 \mathbf{E}_c^{2n} \geq \frac{1}{\Upsilon^2} (\mathcal{A}_c u_c^{2n}, u_c^{2n})_{H_c} + (\mathcal{B}_c \frac{p_c^{2n+1} + p_c^{2n-1}}{2}, \frac{p_c^{2n+1} + p_c^{2n-1}}{2})_{V_c}. \quad (32)$$

In particular

$$\left| \begin{aligned} (\mathcal{A}_c u_c^{2n}, u_c^{2n})_{H_c}^{\frac{1}{2}} &\leq \sqrt{2} \Upsilon \sqrt{\mathbf{E}_c^{2n}}, \\ (\mathcal{B}_c \frac{p_c^{2n+1} + p_c^{2n-1}}{2}, \frac{p_c^{2n+1} + p_c^{2n-1}}{2})_{V_c}^{\frac{1}{2}} &\leq \sqrt{2} \sqrt{\mathbf{E}_c^{2n}}. \end{aligned} \right. \quad (33)$$

We use again the second equation of (19) to obtain

$$(\mathcal{B}_c p_c^{2n\pm 1}, \tilde{p}_c^h)_{V_c} = (\mathcal{B}_c \frac{p_c^{2n+1} + p_c^{2n-1}}{2}, \tilde{p}_c^h)_{V_c} + \Delta t (\mathcal{D}_c u_c^{2n}, \tilde{p}_c^h)_{V_c}.$$

Taking  $\tilde{p}_c^h = p_c^{2n\pm 1}$ , using the Cauchy-Schwarz inequality and the abstract CFL condition on  $\Omega_c$  we obtain

$$\begin{aligned} (\mathcal{B}_c p_c^{2n\pm 1}, p_c^{2n\pm 1})_{V_c} &\leq (\mathcal{B}_c p_c^{2n\pm 1}, p_c^{2n\pm 1})_{V_c}^{\frac{1}{2}} \left( (\mathcal{B}_c \frac{p_c^{2n+1} + p_c^{2n-1}}{2}, \frac{p_c^{2n+1} + p_c^{2n-1}}{2})_{V_c}^{\frac{1}{2}} + \right. \\ &\quad \left. (\mathcal{A}_c u_c^{2n}, u_c^{2n})_{H_c}^{\frac{1}{2}} \right), \end{aligned}$$

and so

$$(\mathcal{B}_c p_c^{2n\pm 1}, p_c^{2n\pm 1})_{V_c}^{\frac{1}{2}} \leq \sqrt{2} (1 + \Upsilon) \sqrt{\mathbf{E}_c^{2n}}. \quad (34)$$

It is clear then that the first equation on (33) and (34) imply the first inequality on (30). In a similar way, using the equations on  $\Omega_f$  we obtain

$$\left| \begin{array}{l} (\mathcal{A}_f u_f^n, u_f^n)_{H_f}^{\frac{1}{2}} \leq \sqrt{2} \Upsilon \sqrt{\mathbf{E}_f^n}, \\ (\mathcal{B}_f p_f^{n\pm \frac{1}{2}}, p_f^{n\pm \frac{1}{2}})_{V_f}^{\frac{1}{2}} \leq \sqrt{2} (1 + \Upsilon) \sqrt{\mathbf{E}_f^n}, \end{array} \right. \quad (35)$$

that implies the second inequality on (30) for the even time steps. Finally we must bound  $u_f^{2n+1}$  in terms of the energy at the even time steps. Subtracting two consecutif time steps of the third equation of (19) we obtain

$$(\mathcal{A}_f u_f^{2n+1}, \tilde{u}_f^h)_{H_f} = (\mathcal{A}_f \frac{u_f^{2n+2} + u_f^{2n}}{2}, \tilde{u}_f^h)_{H_f} - \frac{\Delta t}{2} (\mathcal{D}_f \tilde{u}_f^h, p_f^{2n+\frac{1}{2}} - p_f^{2n+\frac{3}{2}})_{V_f}.$$

We take  $\tilde{u}_f^h = u_f^{2n+1}$ , we use the Cauchy-Schwarz inequality and the second inequality in (29) (CFL condition on  $\Omega_f$ ) to conclude that

$$\begin{aligned} (\mathcal{A}_f u_f^{2n+1}, u_f^{2n+1})_{H_f} &\leq (\mathcal{A}_f u_f^{2n+1}, u_f^{2n+1})_{H_f}^{\frac{1}{2}} \left( (\mathcal{A}_f \frac{u_f^{2n+2} + u_f^{2n}}{2}, \frac{u_f^{2n+2} + u_f^{2n}}{2})_{H_f}^{\frac{1}{2}} + \right. \\ &\quad \left. (\mathcal{B}_f (p_f^{2n+\frac{1}{2}} - p_f^{2n+\frac{3}{2}}), p_f^{2n+\frac{1}{2}} - p_f^{2n+\frac{3}{2}})_{V_f} \right), \end{aligned}$$

This implies

$$(\mathcal{A}_f u_f^{2n+1}, u_f^{2n+1})_{H_f}^{\frac{1}{2}} \leq C \Upsilon \left( \sqrt{\mathbf{E}^{2n}} + \sqrt{\mathbf{E}^{2n+2}} \right),$$

and the proposition is proven.  $\square$

## 7 Error analysis: the main results

We need here additional assumptions. First, we assume that there are spaces

$$\left| \begin{array}{l} Y_l \subset X_l, \quad W_l \subset V_l, \quad l \in \{c, f\}, \\ Y := Y_c \times Y_f, \quad W := W_c \times W_f, \end{array} \right.$$

equipped with the norms  $\|\cdot\|_{Y_l}$ ,  $\|\cdot\|_{W_l}$  and

$$\left| \begin{array}{l} \|(u_c, u_f)\|_Y^2 := \|u_c\|_{Y_c}^2 + \|u_f\|_{Y_f}^2, \quad \forall (u_c, u_f) \in Y, \\ \|(p_c, p_f)\|_W^2 := \|p_c\|_{W_c}^2 + \|p_f\|_{W_f}^2, \quad \forall (p_c, p_f) \in W, \end{array} \right.$$

such that

$$\left| \begin{aligned} \inf_{\tilde{u}_l^h \in X_{l,h}} \|u_l - \tilde{u}_l^h\|_{X_l} &\leq \mathcal{C} h^k \|u_l\|_{Y_l}, \quad \forall u_l \in Y_l, \\ \inf_{\tilde{p}_l^h \in V_{l,h}} \|p_l - \tilde{p}_l^h\|_{V_l} &\leq \mathcal{C} h^k \|p_l\|_{W_l}, \quad \forall p_l \in W_l, \end{aligned} \right|, l \in \{c, f\}, \text{ with } k \geq \frac{1}{2}. \quad (36)$$

In the applications,  $Y_l$  and  $W_l$  are spaces of sufficiently regular functions and  $k$  represents the order of the finite elements.

We also we assume that the following two hypothesis are satisfied

$$\left| \begin{aligned} &\text{There exists a constant } \mathcal{C} > 0 \text{ such that } \quad \forall u_l^h \in X_{l,h}, \quad \|u_l^h\|_{X_l} \leq \frac{\mathcal{C}}{h} \|u_l^h\|_{H_l}, \quad (37) \\ &\text{There exists a constant } \mathcal{C} > 0 \text{ such that} \\ &\quad \left\{ \begin{aligned} &\forall (u_c, u_f^h) \in Y_c \times X_{f,h}, \quad c(u_c, u_f^h) \leq \frac{\mathcal{C}}{\sqrt{h}} \|u_c\|_{Y_c} \|u_f^h\|_{H_f}, \quad (a) \\ &\forall (u_c^h, u_f) \in X_{c,h} \times Y_f, \quad c(u_c^h, u_f) \leq \frac{\mathcal{C}}{\sqrt{h}} \|u_c^h\|_{H_c} \|u_f\|_{Y_f}. \quad (b) \end{aligned} \right. \end{aligned} \right| \quad (38)$$

The first hypothesis can be interpreted as an inverse inequality. As  $\mathcal{D}_l$ ,  $l \in \{c, f\}$  are first order differential operators in space, the definition of the norm  $\|\cdot\|_{X_l}$  given in (2) ensures that (37) will be satisfied if we have uniformly regular meshes on each domain. The presence of the factor  $h^{-1/2}$  in the second hypothesis may seem strange for the abstract problem. It is justified by the applications for which proving (38) is equivalent to proving discrete trace estimates<sup>6</sup>.

Finally we will also assume that we have the following inclusion

$$\mathcal{D}_l X_{l,h} \subset V_{l,h}, \quad (39)$$

This assumption is satisfied in particular by a large class of mixed finite elements. For technical reasons related to the proof of the main result we assume that the initial conditions are zero and that the external forces are regular enough and such that

$$\text{supp}[(f_c, f_f), (g_c, g_f)] \subset (0, T) \times (\Omega_c \cup \Omega_f). \quad (40)$$

Let us introduce the errors:

$$\left| \begin{aligned} e_{u,c}^{2n} &:= u_c(2n \Delta t) - u_c^{2n}, \\ e_{u,f}^n &:= u_f(n \Delta t) - u_f^n, \\ e_{p,c}^{2n+1} &:= p_c((2n+1) \Delta t) - p_c^{2n+1}, \\ e_{p,f}^{n+\frac{1}{2}} &:= p_f((n+\frac{1}{2}) \Delta t) - p_f^{n+\frac{1}{2}}, \end{aligned} \right|$$

and, for each  $T > 0$  the following norms of the errors

$$\left| \begin{aligned} \|(e_{u,c}, e_{u,f})\|_{L_T^\infty(H)} &:= \sup_{t^{2n+\frac{3}{2}} \leq T} (\|e_{u,c}^{2n}\|_{H_c} + \|e_{u,f}^{2n}\|_{H_f} + \|e_{u,f}^{2n+1}\|_{H_f}), \\ \|(e_{p,c}, e_{p,f})\|_{L_T^\infty(V)} &:= \sup_{t^{2n+\frac{3}{2}} \leq T} (\|e_{p,c}^{2n+1}\|_{V_c} + \|e_{p,f}^{2n+\frac{1}{2}}\|_{V_f} + \|e_{p,f}^{2n+\frac{3}{2}}\|_{V_f}). \end{aligned} \right.$$

Finally, for each integer  $l \geq 0$  we define the space

$$C_T^l(Y \times W) := C^l([0, T], (Y_c \times Y_f) \times (W_c \times W_f)).$$

**Theorem 7.1** *Let  $h$  and  $\Delta t$  be constants such that (29) is satisfied. Assume that  $((u_c, u_f), (p_c, p_f))$ , the solution of the continuous problem belongs to*

$$C_{T+l\Delta t}^{\max(3,l)}(Y \times W),$$

where  $T > 0$  is a real number and  $l \geq 0$  is an integer. We will also assume that the discrete initial conditions are second order approximations of the continuous ones. Then we have the following estimate:

$$\left| \begin{aligned} \|(e_{u,c}, e_{u,f})\|_{L_T^\infty(H)} + \|(e_{p,c}, e_{p,f})\|_{L_T^\infty(V)} &\leq \mathcal{C} (1 + T) h^{\min(2,k)} \|(u_c, u_f), (p_c, p_f)\|_{C_T^3(Y \times W)} + \\ &\quad \mathcal{C} T^* h^{\min(\frac{3}{2}-\frac{1}{2l}, k)} \|(u_c, u_f), (p_c, p_f)\|_{C_{T^*}^{l+1}(Y \times W)} \end{aligned} \right|$$

where  $T^* = T + l\Delta t$ .

**Proof:** We refer to<sup>6,7</sup> for the proof and give here the main steps. The first term in the previous estimate comes from the error due to the initial conditions and the interior schemes. The second one is due to the coupling equations. The proof combines energy techniques and boot-strap arguments, as in<sup>3</sup>:

- (i) by energy estimates we first get a  $O(h^{1/2})$  estimate,
- (ii) by induction, one passes from  $O(h^{3/2-1/2^l})$  to  $O(h^{3/2-1/2^{l+1}})$ .

Point (ii) uses successive time discrete derivatives of the error which require additional time regularity on the exact solution.  $\square$

**Remark 7.1** *Numerical experiments suggest that the estimate of theorem 7.1 is optimal<sup>6</sup>.*

**Remark 7.2** *A post treatment in time of the solution allows us to restore the  $O(h^2)$  accuracy<sup>6</sup>.*

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